Broadly speaking, my interests lie in the interplay between combinatorics and topology. I particularly like matroids, hyperplane arrangements, and cellular structures. The research I have done so far demonstrates a strong connection between the topology of orbit spaces and matroid theory. Specifically, the Tutte polynomials of certain matroids can be used to determine the Poincaré polynomial of a quotient space S^{2n-1}/G where G is an abelian subgroup of the orthogonal group $O(2n)$.

1 Background and Motivation

Consider an action of a compact group G on a Reimannian manifold M by isometries. The quotient space of such an action are of interest in many fields of mathematics, but describing their structure can be rather difficult. For example, under what circumstances is M/G still a topological manifold?

Let x be a point in M and consider S_x, the unit sphere in the tangent space to M at x. The isotropy subgroup $G_x \subseteq G$ that fixes x induces an action on the unit tangent sphere. An excision argument shows that the space M/G cannot be a manifold unless the orbit space of the unit sphere, S_x/G_x, is a homology sphere for every $x \in M$. Knowing the homology of quotient spaces of the form S_x/G_x is therefore an important step in understanding the structure of M/G. However, determining the topology of S_x/G_x can be challenging, especially when the action is not free. For instance, it is not known under what circumstances this quotient space will be a homology sphere.

The above discussion demonstrates one motivation to study quotients of spheres by isometries. Let G be a subgroup of the orthogonal group $O(n)$ act on a sphere S^{n-1} with orbit space X. If G is abelian, this action becomes much easier to describe. This is because we can simultaneously diagonalize all of the elements of G over $\mathbb{C}$. This diagonalization yields a subgroup of $O(n)$ conjugate to G, whose action on S^{n-1} yields a orbit space isometric to S^{n-1}/G. We can therefore assume that the action of a finitely generated abelian G on X be described by a list of diagonal matrices over $\mathbb{C}$. For this reason, we will focus on the case where G is abelian.

The study of these quotient spaces will require some background in combinatorics. In particular, matroids will play a key role in the results that follow. Matroids were first invented by Whitney in 1935 as a way to generalize linear independence [5].

Definition 1.1. A matroid M is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}(M) \subseteq \mathcal{P}(E)$ denotes the independent subsets of E. The independent subsets $\mathcal{I}(M)$ respect the following axioms:

1) $\emptyset \in \mathcal{I}$

2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$

3) If $A, B \in \mathcal{I}(M)$ and $|A| > |B|$, then $\exists e \in A$ such that $e \notin B$ and $B \cup e \in \mathcal{I}$

Matroids can be naturally associated to a number of mathematical structures. For example, the edges of a graph form a matroid where sets of edges are independent if and only if they contain no cycles. A matroid also arises from every matrix A where $E = \{\text{columns of } A\}$ and subsets of E are independent if and only if they are linearly independent. A matroid is *representable* over a field F if it can be expressed as the columns of a matrix over F in this way.

The Tutte polynomial is a powerful matroid invariant. It is defined recursively by the operations of deletion $M - e$ and contraction M/e in matroids (analogous to deletion and contraction in graphs). In the following definition, a *loop* of a matroid is an edge that is in no independent set. A *coloop* is an element found in every maximal independent set.

Definition 1.2. The Tutte Polynomial of the matroid M, denoted $T(M; x, y)$, is the unique two-variable polynomial satisfying:

1) If M is the matroid of a single coloop, then $T(M; x, y) = x$

2) If M is the matroid of a single loop, then $T(M; x, y) = y$

2) If $e \in M$ is a loop or coloop, $T(M; x, y) = T(e; x, y)T(M - e; x, y)$

3) If e is neither, $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$

Tutte polynomials have a variety of import applications. For example, they can be used to determine the chromatic polynomial of a graph, compute number of nowhere zero flows on an oriented graph, and count the number of regions in a hyperplane arrangement.

2 Quotients by Cyclic Groups

What do quotients of spheres by cyclic groups look like? A familiar example of such a quotient is a lens space $S^{2n-1}/\mathbb{Z}_k$. Lens spaces are often referenced in an introductory algebraic topology class as a cautionary counterexample: there are large classes of lens spaces that share the same homology and fundamental group but are not homotopically equivalent. One way to describe a lens space is to consider the decomposition of an odd dimensional sphere into the join of circles $S^{2n-1} \cong S^1_1 * \cdots * S^1_n$. The generator of $G \cong \mathbb{Z}_m$ acts by a rotation on each S^1_i in the join. Specifically, the generator rotates S^1_1 by $\frac{2\pi}{k_1}$, S^1_2 by $\frac{2\pi}{k_2}$, etc. In order to be a lens space, it is furthermore required that $\gcd(k_i, m) = 1$ for all $1 \leq i \leq [n]$. The homology and homotopy classes of lens spaces are very well understood [1]. While the gcd restriction forces the lens space to be a manifold, none of the lens spaces are homology spheres.

After learning about lens spaces, a natural question to ask would be: what if we ignore this gcd requirement? Since G is cyclic, the action can be described by a single matrix that generates G. We can demand that this matrix be diagonalized over $\mathbb{C}$, then we can see that the action is described by rotations of certain eigencircles as in the case of a lens space above. (Note: in order to use this geometric interpretation of the actions, we are assuming in these cases that $\mathbb{Z}_2$ actions preserve orientation). The cohomology rings of these generalized lens spaces were found by Stephen Willson [6]. Interestingly, his results about these spaces extended to quotients of homology spheres, rather than just actions on metric spheres.

3 Quotients by Elementary Abelian p-Groups

In 1999, Ed Swartz discovered some interesting connections between quotients of spheres and matroid theory [4]. Specifically, he classified the homology of quotients of spheres by subgroups of $O(2n)$ isomorphic to $(\mathbb{Z}_p)^r$ for p an odd prime, and quotients by subgroups of $O(n)$ by $(\mathbb{Z}_2)^r$. Note that if p is odd, every quotient $S^{2n}/(\mathbb{Z}_p)^r$ is the suspension of a quotient $S^{2n-1}/(\mathbb{Z}_p)^r$, so it suffices to study orbit spaces of odd-dimensional spheres. The diagonalized matrices corresponding to the generators of $(\mathbb{Z}_p)^r$ can be used to form a $r \times n$ matrix that described the action completely. Each row of this matrix corresponds to the action of a single generator; each column corresponds to the action of the group on a single circle in the join.

The matroid represented by this matrix became the main object of study. Recall that the matroid associated to a matrix is the set of columns along with the independence relation given by linear independence. The abstraction of the matrix to the matroid eliminates some unnecessary data that does not affect the isometry class of the quotient: e.g. choice of generators for G, ordering of basis of $\mathbb{R}^n$, etc. In this way, we can see that the matroid is an invariant of the orbit space. Swartz used the Tutte polynomial of this matroid to calculate the homology of the quotient $S^{2n-1}/(\mathbb{Z}_p)^r$ as well as the homology of the regular part of such a quotient, i.e. the image of the set on which $(\mathbb{Z}_p)^r$ acts freely.

4 Quotients by Real Tori

After such success with finite tori $\mathbb{Z}_p^r$, we might suspect that quotients of spheres by topological subgroups of $O(2n)$ isomorphic to real tori T^r may yield similarly interesting results. The homology of the regular part of such a quotient was already known to be related to the Tutte Polynomial of an associated matroid. I pursued this line of thinking, and was able to compute the integral homology of the orbit space of any linear action of T^r on S^{2n-1}. Let $\widetilde{P}_X(t)$ denote the reduced Poincaré polynomial: $\widetilde{P}_X(t) = \sum_{i=0}^{n} \dim(H_i(X;\mathbb{Q}))\, t^i$

Theorem 4.1. *Let $X = S^{2n-1}/T^r$ with associated matroid M_X. Then $\widetilde{P}_X(t) = t^{r-1}T(M_X; 0, t^2)$, the one-variable Tutte polynomial. Furthermore, $H_i(X;\mathbb{Z}) \cong H_i(X;\mathbb{Q})$, i.e. $H_i(X;\mathbb{Z})$ has no torsion.*

The *rational singular set* of an orbit space of T^r is the image of the points whose isotropy groups are infinite. Since the homology of the rationally free part of the action had already been found in the context of toric topology [3] , we wondered if the singular set could have interesting properties as well. I was able to show that the singular set, being the image of certain subspheres of S^{2n-1}, was an arrangement. Furthermore, the lattice of this arrangement is the order dual of the lattice of flats of the associated matroid. Using the results on the topology of arrangements in [7], I was able to show that the Poincaré polynomial of the rational singular set was the difference of two Tutte polynomials:

Theorem 4.2. *Let $\mathcal{S}$ denote the rational singular set of the orbit space S^{2n-1}/T^r. Then $\widetilde{P}_{\mathcal{S}}(t) = t^{r-2}(T(M_X; 1, t^2) - T(M_X; 0, t^2))$. Furthermore, $H_i(\mathcal{S}; \mathbb{Z}) \cong H_i(\mathcal{S}; \mathbb{Q})$, i.e. $H_i(\mathcal{S}; \mathbb{Z})$ has no torsion.*

5 Other Finite Abelian Groups

Recently, I have turned my studies toward other finite abelian groups G. The findings of Willson and Swartz discussed earlier can be used to compute the homology of any orbit space S^{2n-1}/G when G is a cyclic group or a elementary abelian p-group. However, they do not describe the topology of a quotient of a sphere by an action of any other group of prime power order, such as $\mathbb{Z}_4 \times \mathbb{Z}_4$. Combining the results on cyclic groups and p-groups into a cohesive theory has proven to be quite challenging.

Since any finite abelian group is a direct sum of groups of prime power order, it is reasonable to first consider groups G where the order of G is a prime power. In approaching this problem, it has been necessary to construct a sequence of matroids, one for each prime power that appears as the order of an element of G. Evidence suggests that this sequence of matroids completely describe the homology of the quotient space. Based on preliminary results, there is reason to suspect that a solution to this problem could be readily extended to understanding quotients of spheres by all finite abelian groups.

For a group G with exponent p^k, we define a sequence of matroids as follows. First, find a matrix describing the action as before by using the diagonalized basis elements as rows. If a row corresponds to a generator of order p^a, multiply this row by p^{k-a}. Then, rewrite the entries of this matrix mod p^k. We wish for M_{p^k} to be a matroid associated to this matrix, but this is slightly difficult since $\mathbb{Z}_{p^k}$ is not a field. To define a rank function on the columns, consider them as elements of G. Let $A \subseteq G$ be the subgroup generated by a subset of the columns, then define the rank of that subset to be the rank of A tensored with $\mathbb{Z}_p$. Details regarding this notion of independence on elements of a group can be found in [**?**].

Using this technique, we can get a sequence of matroids $M_{p^k}, M_{p^{k-1}}, \ldots, M_p$, arising from the new matrix of the action rewritten modulo $p^k, p^{k-1}, \ldots, p$. These matroids have weak maps between them (i.e. each new matroid may have more dependencies than the last). The matroid $2M$ in the conjecture below denotes the doubled matroid of M, where each element in M is given a parallel element (i.e. every column is written twice).

Conjecture 1. *Consider the following sequence of Tutte polynomials generated by the matrix corresponding to the quotient space X:*

$$t^{r-1}T(2M_{p^k}; 0, t), t^{r-1}T(2M_{p^{k-1}}; 0, t), \cdots t^{r-1}T(2M_p; 0, t).$$

If a copy of t^i appears in m, but not $m+1$, consecutive polynomials of this sequence, it corresponds to a $\mathbb{Z}_{p^m}$ summand in $H_i(X; \mathbb{Z}_{p^k})$. Every summand of $H_i(X; \mathbb{Z}_{p^k})$ can be found in this manner.

Example 5.1. Consider the quotient of S^7 by $\mathbb{Z}_4 \times \mathbb{Z}_4$ represented by: $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, then
$t^{r-1}T(2M_4; 0, t) = t^7 + 2t^6 + 3t^5 + 4t^4 + 5t^3 + 2t^2$

$t^{r-1}T(2M_2; 0, t) = t^7 + 2t^6 + 3t^5 + 3t^4 + 3t^3 + t^2$

Copies of t^i appearing in both polynomials become $\mathbb{Z}_4$ summands in $H_i(X; \mathbb{Z}_{p^k})$, whereas the extra $t^4, 2t^3$, and t^2 in $t^{r-1}T(2M_4; 0, t)$ become $\mathbb{Z}_2$ summands in $H_i(X; \mathbb{Z}_{p^k})$:

$H_1(X; \mathbb{Z}_4) = 0$
$H_2(X; \mathbb{Z}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$
$H_3(X; \mathbb{Z}_4) = (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_4)^3$
$H_4(X; \mathbb{Z}_4) = \mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3$
$H_5(X; \mathbb{Z}_4) = (\mathbb{Z}_4)^3$
$H_6(X; \mathbb{Z}_4) = (\mathbb{Z}_4)^2$
$H_7(X; \mathbb{Z}_4) = \mathbb{Z}_4$

It is interesting to note that M_4 in the above example is $U_{2,4}$, the matroid in which any pair of elements is independent and any three elements are dependent. This matroid cannot be represented by a matrix over $\mathbb{Z}_2$, but its structure and Tutte polynomial are part of an action by $\mathbb{Z}_4 \times \mathbb{Z}_4$.

6 Directions for Further Research

The sequence of matroids being associated to these spaces may be an interesting object of study, independent of its topological connections. In the $(Z_p)^r$ case, the matroids came from the linear independence relation on the columns of a $\mathbb{Z}_p$-matrix. However, the matroids in the new sequence come from matrices over different powers p^k. I would be interested in learning more about the class of matroids “representable” by matrices with entries modulo p^k, and the weak maps that go between them.

In addition to completing the proof of the above conjecture, there are a number of directions for further research on these quotient spaces. For example, the singular and free sets corresponding to these more general actions could be studied. Right now, very little is now known about quotients S^n/G when G is not abelian. It may be fruitful to study the cases where G is a dihedral or symmetric group.

During the course of my studies, I have also had the opportunity to learn about hyperplane arrangements, h-vectors, and oriented matroids. Working in these fields further would take advantage of my background in both topology and matroid theory. The study of polyhedra and other objects in discrete geometry would also marry geometric intuition and combinatorial methods. There are furthermore a number of open problems in graph theory that relate to topology. I look forward to pursuing these other interests more in the future.

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